Viscous Interaction of Flow Redevelopment after Flow Reattachment with Supersonic External Streams

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A flow model has been developed to study the flow development after reattachment with supersonic external streams. Special attention is given to the pressure difference across the viscous layer, and it is suggested that such a flow redevelopment can be treated as a relaxation of this pressure difference. Upon correlating the pressure difference with a slope parameter of the velocity profile, the system of equations governing the flow would produce a saddle point singularity corresponding to the fully rehabilitated asymptotic flow condition. A method of calculation for this flowfield, in conjunction with the matching of the upstream flow, has been derived and is discussed. Samples of calculations are also presented. Reasonably good agreement with experimental data has also been observed.

Nomenclature

$A_i,B_i,C_i,D_i,$ $(i=1,2,3)$ = coefficient functions
C = Crocco number, $C=u/V_m$
E,F,G = functions
f_1,f_2,f_3,f_4 = integral quantities
H = step height
h_b = backflow height within wake region
k = proportional constant
ℓ_m = length of upstream constant pressure jet mixing region
M = Mach number
P = pressure
S = slope parameter
u,v = velocity component in main flow and normal flow directions
x,y = coordinates
V_m = maximum speed
ρ = density
β = angle of streamline of viscous flow
δ = thickness of viscous layer within redeveloping region
δ_a,δ_b = thickness of viscous layer above and below dividing streamline within recompression region
ζ = dimensionless coordinate
τ = shear stress
ϕ = dimensionless velocity within viscous layer
γ = ratio of specific heats
Δ = determinant defined in text
δ^{**} = momentum thickness of boundary layer of flow approaching the base
θ_r = reattachment angle (a positive quantity) also indicating the path of dividing streamline
ϵ = eddy diffusivity
σ = spread rate parameter

Subscripts

e = freestream
b = base or backflow within the recompression region
w = wall
R,R' = values at sections R and R', respectively
∞ = asymptotic state
$1,1a$ = flow and freestream flow approaching the base
$2,2a$ = flow and freestream flow after corner expansion
$2m$ = upper edge of mixing layer within the constant pressure region
d = dividing streamline

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Introduction

FOR the flow of a real fluid past a blunt body, the viscous layer attached to the wall cannot cope with the eventual pressure rise and separates from the solid wall forming a wake behind the body. Since the wake pressure is usually lower than its freestream value, a considerable portion of the drag is attributed to this flow separation phenomenon. A large number of research investigations have been associated with the study of separated flow problems. Indeed, the intensive effort carried out within the last two decades within the high-speed flow regime has led to a much better understanding of these problems. It is generally recognized that after the flow separates from the wall, a mixing process (usually turbulent) under an essentially constant pressure condition occurs along the early part of the wake boundary. Near the end of the wake, the flow either reattaches to a solid wall or realigns itself to the original flow direction, initiating a recompressive process. Part of the fluid entrained within the viscous layer is turned back to form the recirculatory wake flow, while the remaining fluid is allowed to proceed downstream. It is thus clear that the mutual interaction between the viscous and inviscid streams controls and influences the solution to the problem. While the viscous flows are more or less guided by the corresponding inviscid streams, the configuration of the inviscid flow, as well as the level of the constant base pressure prevailing within the major portion of the wake, are influenced by the viscous flow mechanisms.

Since these interaction mechanisms were pointed out by Crocco and Lees,[1] research on separated flows within the high-speed flow regime has been pursued along two major avenues. The Chapman-Korst model[2,3] emphasizes the study and delineation of the individual flow components of the separated flows when they can be properly identified and the flow solutions are reached by integrating these individual components and considerations. Following the original idea of Crocco and Lees,[1] Lees and Reeves[4] considered these interactions from an integral approach by treating the attached and separated flows under one single framework. Upon supplementing the basic conservation principles with the moment of momentum relationship, they showed that for separated flows, a critical point exists immediately downstream of the wake. The flow solution for a particular problem is obtained when the flow passes smoothly through this critical point. In restricting itself to small perturbations of the approaching freestream flow conditions, this approach has yielded good results for laminar base flow problems.

It should be pointed out that although the adoption of the moment of momentum equation to the study of laminar separated flows has met considerable success by Lees and Reeves, it certainly is not the rule that should be followed for any problem of a similar nature. Alber and Lees[5] employed the same approach for supersonic flow past a backstep in the turbulent flow regime. The base pressures obtained from their calculations are too high and do not agree with the experimental data. Experimental observations under these conditions showed that for turbulent flow with high Reynolds numbers, the base pressure is low and the main flow would be directed toward the lower wall with a relatively large angle prior to flow recompression. As a result of the streamline curvature effect, a considerable difference of pressure exists across the viscous layer in the neighborhood of the point of reattachment. This evidence clearly indicates the inadequacy of the ordinary boundary layer analysis for these problems.

The least explored flow components in the Chapman-Korst model are recompression, reattachment, and redevelopment of the flow. The recompression along the dividing streamline, originally suggested by Korst is isentropic, was later modified through the introduction of "recompression parameter," "recompression pressure ratio" by Nash,[6] Addy,[7] or "reattachment correlation" by Page.[8] Although these corrections or correlations are useful in providing simple working relations to determine base pressures, it would be of utmost interest to investigate these flow processes in detail. Indeed, recent attempts to study these flowfields[9-11] have shown that it is possible to evaluate these flow processes in a more satisfactory manner. It was shown[9] that by linking the dividing streamline velocity together with its slope of the profile, it is possible to calculate the recompression process up to the point of reattachment. The pressure differences across the viscous layers were also accounted for. This method of calculations has also been extended to study laminar flows[10] and low-speed flows.[11]

The present study specifically considers the component of flow redevelopment after flow reattachment within the turbulent flow regime. Previous study by McDonald[12] treated this problem as an adjustment of the form factor of the velocity profile within the scope of the boundary layer concept. As the pressure difference across the viscous layer is not negligible in the vicinity of the point of reattachment, particular attention is given here to this pressure difference. It is also recognized that ordinary boundary layer flow with no traverse pressure difference should prevail at "far downstream" positions. It may thus be conceived that this process of flow redevelopment can be interpreted as a process of relaxation of the pressure difference acorss the viscous layer. Once this particular character is identified, it may be advantageously exploited for the analysis of the flow. It is intended here to show the employment of this relaxation phenomenon to the analysis of such a flow process of redevelopment. It shall become obvious that this analysis refers to a small physical region downstream of the point of reattachment even when the fully rehabilitated state shall be labelled mathematically as "asymptotic condition" at "far downstream positions."

Theoretical Consideration

Fundamental Equations

Referring to Fig. 1, where the physical region of flow redevelopment after reattachment is shown, one may write the continuity, streamwise, and transverse momentum equations for the region downstream of R', respectively, as

$$\partial(\rho u)/\partial x+\partial(\rho v)\partial y = 0 \tag{1}$$

$$\rho u(\partial u/\partial x)+\rho v(\partial u/\partial y) = -\partial p/\partial x+\partial \tau_{yx}/\partial y \tag{2}$$

$$\rho u(\partial v/\partial x)+\rho v(\partial v/\partial y) = -\partial p/\partial y+\partial \tau_{xy}/\partial x \tag{3}$$

Isoenergetic flowfield is assumed throughout the flow so that consideration of the energy equation is conveniently eliminated. One may integrate these equations across the viscous layer to obtain

$$\tan\beta_e=\frac{v_e}{u_e}=\frac{d}{dx}\left[\delta\int_0^1\left(1-\frac{\rho}{\rho_e}\frac{u}{u_e}\right)d\zeta\right] - \frac{\delta\int_0^1\frac{\rho}{\rho_e}\frac{u}{u_e}d\zeta}{\rho_e u_e}\frac{d(\rho_e u_e)}{dx} \tag{4}$$

$$\tau_w=\frac{d}{dx}\left[\rho_e u_e^2\delta\int_0^1\frac{\rho}{\rho_e}\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)d\zeta\right] +\delta\int_0^1\left(1-\frac{\rho}{\rho_e}\frac{u}{u_e}\right)d\zeta\,\rho_e u_e \cdot\frac{du_e}{dx}-\frac{d}{dx}\left[p_e\delta\int_0^1\left(\frac{p}{p_e}-1\right)d\zeta\right] \tag{5}$$

and

$$\left(\frac{p_w}{p_e}-1\right)=\frac{\rho_e u_e^2}{p_e}\left(\tan^2\beta_e-\tan\beta_e\frac{d\delta}{dx}\right)+\frac{1}{p_e}\frac{d}{dx} \cdot\left[\rho_e u_e^2\delta\int_0^1\frac{\rho}{\rho_e}\left(\frac{u}{u_e}\right)^2\left(\frac{v}{u}\right)d\zeta\right] -\frac{1}{p_e}\frac{d}{dx}\left[\delta\int_0^1\tau_{xy}d\zeta\right] \tag{6}$$

where $\zeta=y/\delta$, and x,δ have been normalized by any reference length, e.g., step height H. Upon introducing $C_e=u_e/V_m$, $\phi=u/u_e$, assuming

$$\frac{p}{p_e}=\frac{p_w}{p_e}-\frac{3}{2}\left(\frac{p_w}{p_e}-1\right)\zeta+\tfrac{1}{2}\left(\frac{p_w}{p_e}-1\right)\zeta^2 \tag{7}$$

$$\tan\beta=v/u=\tan\beta_e(2\zeta-\zeta^2) \tag{8}$$

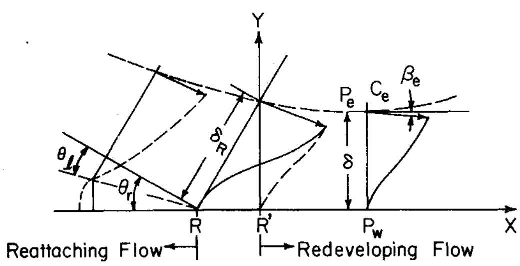

Fig. 1 Flow redevelopment after reattachment.

and neglecting the wall shear stress in Eq. (5) and the rate of change of the shear stress integral in Eq. (6) in this flow region, one obtains

$$\tan\beta_e = \frac{d}{dx}[\delta(1-f_1)] - \frac{\delta f_1}{(1-C_e^2)^{1/(\gamma-1)}C_e}\frac{d}{dx}\times[(1-C_e^2)^{1/(\gamma-1)}C_e] \quad (9)$$

$$\frac{d}{dx}\left[(1-C_e^2)^{(1/\gamma-1)}C_e^2\delta f_2\right] + (1-C_e^2)^{(1/\gamma-1)}C_e\delta(1-f_1)\frac{dC_e}{dx} - \frac{3}{16}\frac{\gamma-1}{\gamma}\frac{d}{dx}\left[(1-C_e^2)^{(\gamma/\gamma-1)}\delta(\frac{p_w}{p_e}-1)\right]=0 \quad (10)$$

and

$$\frac{\gamma-1}{2\gamma}(\frac{p_w}{p_e}-1) = \frac{C_e^2}{1-C_e^2}(\tan^2\beta_e - \tan\beta_e\frac{d\delta}{dx}) + \left[(1-C_e^2)^{(\gamma/\gamma-1)}\right]^{-1}\cdot\frac{d}{dx}\left[(1-C_e^2)^{(1/\gamma-1)}C_e^2\delta f_3\right] \quad (11)$$

where

$$f_1 = \int_0^1 \frac{\rho}{\rho_e}\phi\, d\zeta, \quad f_2 = \int_0^1 \frac{\rho}{\rho_e}\phi(1-\phi)\, d\zeta$$

$$f_3 = \tan\beta_e\int_0^1 \frac{\rho}{\rho_e}\phi^2(2\zeta-\zeta^2)\, d\zeta$$

It should be mentioned here that disregarding the wall shear stress within this region is entirely justified. It is shown that vanishing wall shear stress must occur at the point of reattachment even under turbulent flow conditions. It will be seen later that this recompression and redevelopment process will be completed shortly after reattachment (on the order of few step heights), that the physical region under consideration is indeed small, and thus, the wall shear stress could not have significantly influenced the flow phenomenon. In addition, early calculations on laminar flows[10] showed that the wall skin friction within this region has never overshot the equivalent unseparated skin friction level and is indeed smalll, while the insertion of the corresponding equilibrium turbulent skin friction according to Spalding and Chi[13] into these turbulent flow calculations has made negligible influence on the results.

For the same reason of the small physical region under consideration, the lateral shear integral could not have changed significantly; so that its influence on the pressure difference across the viscous layer can be disregarded. This is true even when the end of this region shall be termed mathematically as "far downstream positions." It should be stressed that ignoring the wall shear stress as well as the rate of change of the lateral shear stress integral does not imply that the dissipation process has been entirely disregarded. In fact, as it will be discussed in the following section, the existence of the shear stress (and thus dissipation) within the fluid is of paramount importance to effect the completion of this redevelopment-recompression process.

Velocity Profile and Correlation of the Slope Parameter to the Pressure Difference Across the Viscous Layer

For a simple representation of the flow within the viscous layer, one may select the velocity profile as given by

$$\phi = S\zeta + (3-2S)\zeta^2 + (S-2)\zeta^3 \quad (12)$$

which obviously satisfies the conditions of $\phi=0$ at $\zeta=0$ and $\phi=1$, $\partial\phi/\partial\zeta=0$ at $\zeta=1$. $S(S=\partial\phi/\partial\zeta|_{\zeta=0})$ is a slope parameter that will be correlated with the pressure difference across the viscous layer through the following considerations.

After the flow reattaches to the wall, the main flow turns continuously toward the horizontal direction and the pressure rises steadily toward the original freestream value. The viscous layer, especially the part in the proximity of the wall, can follow such a continuous increase in pressure only when a continuous chain of transfer of mechanical energy through shear work is effected. It can be shown from basic dynamic principle that a net gain of mechanical energy (and thus coping with the pressure rise) is possible within the viscous flowfield only if the local curvature of the velocity profile is positive. It would imply that for the present problem of the flow redevelopment, the coefficient of ζ^2 in Eq. (12) should be positive throughout the flow. On the other hand, at far downstream position, this coefficient must vanish as there is no additional pressure rise. Coincidentially, the pressure difference across the viscous layer is the largest immediately after reattachment (section R' in Fig. 1), remains to be positive throughout the region of redevelopment and reduces to zero when the flow is fully rehabilitated. A simple and attractive way to correlate these two outstanding features of the flow within this region would be that

$$3-2S = k[(p_w/p_e)-1] \quad (13)‡$$

where k is a constant that may be evaluated from the initial information at section R'. Such a coupling would assure a continuous chain of transfer of mechanical energy toward the wall, and the fully rehabilitated state is reached only when the pressure difference across the viscous layer is fully relaxed.

It should be mentioned that the slope parameter S may be alternatively coupled with the streamwise pressure gradient as suggested from the compatibility relationship of the boundary layer concept. Indeed, early calculations showed that it is possible to relate S with the wall pressure gradient, and it does not change the character of the flowfield. However, it introduces complications to the calculation procedures, and the results are not significantly different from the present results. Since these reattaching flows always couple with the existence of adverse streamwise pressure gradient as well as the transverse pressure difference across the viscous layer, the simpler and more appealing coupling relation [Eq. (13)] has been adopted.

If one ignores the effect of pressure difference across the layer on the density of the fluid so that for isoenergetic flows, the density ratio may be evaluated from

$$\rho/\rho_e = (1-C_e^2)/(1-C_e^2\phi^2) \quad (14)$$

and the integrals f_1, f_2, and f_3 will only be functions of C_e and p_w/p_e. In addition, the freestream supersonic flow follows also the Prandtl-Meyer relationship given by

$$\frac{d\beta_e}{dC_e} = -\frac{1}{C_e}\left[\frac{2}{\gamma-1}\frac{C_e^2}{1-C_e^2}-1\right]^{1/2} \quad (15)$$

Equations (9) through (11) may now be converted, after considerable algebraic manipulations, into

$$A_1(d\delta/dx) + A_2(dC_e/dx) + A_3[d(p_w/p_e)/dx] = D_1$$

$$B_1(d\delta/dx) + B_2(dC_e/dx) + B_3[d(p_w/p_e)/dx] = D_2$$

$$C_1(d\delta/dx) + C_2(dC_e/dx) + C_3[d(p_w/p_e)/dx] = D_3 \quad (16)$$

‡The term on the right-hand side of this equation may be considered as the leading term of a power series expansion.

where

$$A_1 = 1 - f_1$$

$$A_2 = -\delta\left[\frac{\partial f_1}{\partial C_e} + \frac{f_1}{C_e}\left(1 - \frac{2C_e^2}{(\gamma-1)(1-C_e^2)}\right)\right]$$

$$A_3 = -\delta[\partial f_1/\partial(p_w/p_e)]$$

$$B_1 = f_2 - \frac{3}{16}\,\frac{\gamma-1}{\gamma}\,\frac{1-C_e^2}{C_e^2}\left(\frac{p_w}{p_e} - 1\right)$$

$$B_2 = \delta\left[\frac{2f_2}{C_e}\left(1 - \frac{C_e^2}{(\gamma-1)(1-C_e^2)}\right) + \frac{\partial f_2}{\partial c_e} + \frac{1-f_1}{C_e} + 3/8\,\frac{(p_w/p_e)-1}{C_e}\right]$$

$$B_3 = \delta\left[\frac{\partial f_2}{\partial(p_w/p_e)} - \frac{3}{16}\,\frac{\gamma-1}{\gamma}\,\frac{1-C_e^2}{C_e^2}\right]$$

$$C_1 = [C_e^2/(1-C_e^2)](f_3 - \tan\beta_e)$$

$$C_2 = \delta\left[\frac{C_e^2}{1-C_e^2}\,\frac{\partial f_3}{\partial C_e} + \frac{2f_3C_e}{1-C_e^2}\left(1 - \frac{C_e^2}{(\gamma-1)(1-C_e^2)}\right)\right]$$

$$C_3 = \delta[C_e^2/(1-C_e^2)][\partial f_3/\partial(p_w/p_e)]$$

$$D_1 = \tan\beta_e$$

$$D_2 = 0$$

$$D_3 = \frac{\gamma-1}{2\gamma}\left(\frac{p_w}{p_e} - 1\right) - \frac{C_e^2}{1-c_e^2}\tan^2\beta_e \qquad (17)$$

Since functions f_1, f_2, and f_3 cannot be expressed in closed form, their derivatives with respect to C_e and p_w/p_e can only be obtained through numerical differentiation.

The Asymptotic State as a Saddle Point Singularity

Equation (16) would yield the values of the derivatives as

$$\frac{d\delta}{dx} = \frac{(A_2B_3 - A_3B_2)D_3 + D_1(B_2C_3 - B_3C_2)}{\Delta} \qquad (18a)$$

$$\frac{dC_e}{dx} = \frac{D_3(A_3B_1 - A_1B_3) + (B_3C_1 - B_1C_3)D_1}{\Delta} \qquad (18b)$$

$$\frac{d(p_w/p_e)}{dx} = \frac{(A_1B_2 - A_2B_1)D_3 + D_1(B_1C_2 - B_2C_1)}{\Delta} \qquad (18c)$$

where

$$\Delta = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Because of the autonomous character of the system of equations, the variable x can be completely eliminated from the system of equations and one obtains from Eq. (18)

$$\frac{d\delta}{dC_e} = \frac{(A_2B_3 - A_3B_2)D_3 + (B_2C_3 - B_3C_2)D_1}{(A_3B_1 - A_1B_3)D_3 + (B_3C_1 - B_1C_3)D_1} = \delta f_4\left(C_e, \frac{p_w}{p_e}\right) \qquad (19a)$$

$$\frac{d(p_w/p_e)}{dC_e} = \frac{(A_1B_2 - A_2B_1)D_3 + (B_1C_2 - B_2C_1)D_1}{(A_3B_1 - A_1B_3)D_3 + (B_3C_1 - B_1C_3)D_1} \qquad (19b)$$

where f_4 is the residual after δ has been extracted from the expression for $d\delta/dC_e$. Obviously, both D_1 and D_3 vanish at the fully rehabilitated state ($(p_w/p_e) = 1$, $\beta_e = 0$), and this asymptotic condition is a singularity for the system of equations. Since Eq. (19b) is independent of δ, it would be sufficient to investigate this equation. It is shown in the Appendix that it is a singularity of the saddle point type.

Upon evaluating the derivatives dD_3/dC_e and dD_1/dC_e at the asymptotic condition (identified by subscript ∞), one obtains from Eq. (19)

$$\left.\frac{d(p_w/p_e)}{dC_e}\right|_\infty = \frac{-F + (F^2 - 4EG)^{1/2}}{2E} \qquad (20)$$

and

$$\left.\frac{1}{\delta}\,\frac{d\delta}{dC_e}\right|_\infty = \frac{1}{\delta}\;\frac{\dfrac{\gamma-1}{2\gamma}(A_2B_3 - A_3B_2)_\infty \left.\dfrac{d(p_w/p_e)}{dC_e}\right|_\infty + (B_2C_3 - B_3C_2)_\infty \left.\dfrac{d\beta_e}{dC_e}\right|_\infty}{E\left.\dfrac{d(p_w/p_e)}{dC_e}\right|_\infty + (C_1B_3 - B_1C_3)_\infty \left.\dfrac{d\beta_e}{dC_e}\right|_\infty} \qquad (21)\S$$

where

$$E = (\gamma-1)/2\gamma(B_1A_3 - A_1B_3)_\infty$$

$$F = (B_3C_1 - B_1C_3)_\infty \left.\frac{d\beta_e}{dC_e}\right|_\infty - \frac{\gamma-1}{2\gamma}(A_1B_2 - A_2B_1)_\infty$$

$$G = -(B_1C_2 - C_1B_2)_\infty \left.\frac{d\beta_e}{dC_e}\right|_\infty \qquad (22)$$

and

$$\left.\frac{d\beta_e}{dC_e}\right|_\infty = -\frac{1}{C_{e_\infty}}\left[\frac{2}{\gamma-1}\,\frac{C_{e_\infty}^2}{1-C_{e_\infty}^2} - 1\right]^{1/2}$$

according to Eq. (15).

One may now select a parametric value of k for a certain value C_{e_∞} for the asymptotic condition and perform the integration of Eqs. (19a, b) with the initial derivatives given by Eqs. (20) and (21). Such results of integration are presented in Fig. 2. It is interesting to note that the relationship between p_w/p_e and C_e is independent of δ and that δ changes in a manner such that the flow passes through a geometrical throat without encountering any difficulty. To transform the results back to the physical plane, integration of either Eqs. (18a) or (18b) must be performed. It is understood, of course, that the asymptotic condition is reached only at mathematically infinite, but practically finite, x, values. Indeed, it will be observed from the results of calculations that this mathematically far downstream position is only a few step heights downstream of the point of reattachment.

§Note that the right-hand side of this equation in this form is actually independent of δ.

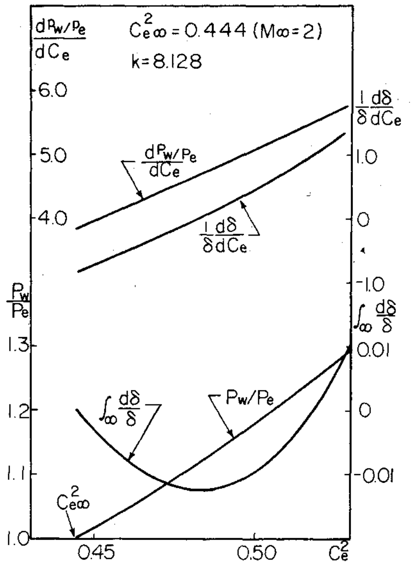

Fig. 2 **Flow properties within redevelopment flow region.**

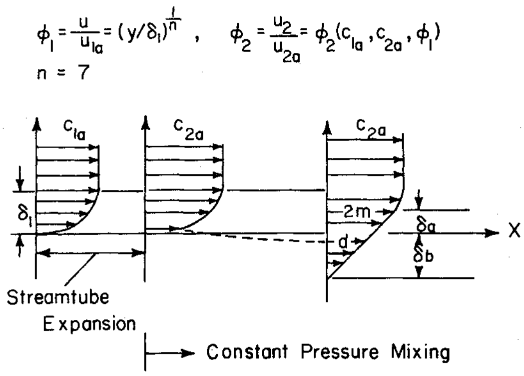

Fig. 3 **Streamtube expansion and constant pressure turbulent jet mixing.**

Matching with the Upstream Reattachment Flow

To assess the influence of this component of flow redevelopment to the overall flowfield, the foregoing analysis and calculations must be joined with the upstream mixing-reattaching flow. Description of the upstream flow and the matching or joining of these flowfields are discussed briefly in this section. The analysis for the upstream recompression flow follows essentially the scheme developed previously,[9] but the analysis for the constant pressure developing turbulent jet mixing process has been completely replaced. They are discussed separately as follows.

1) Expansion at Corner and Ensuing Constant Pressure Developing Turbulent Jet Mixing

It is well known that the initial boundary layer exerts considerable influence to the turbulent jet mixing process and this initial boundary layer results from the expansion of the freestream and its viscous layer approaching the base. Page[14] has observed that the original turbulent intensity within the viscous layer has been reduced considerably as it goes through the expansion region. It would suggest that the turbulent mixing activities will be mainly confined within the new mixing layer along the "jet boundary" and the rest of the original viscous layer will remain unchanged until the effect of the new mixing process is felt. (In fact, as suggested by H. H. Korts, the initial part of the new mixing region may be laminar until transition occurs.) If one stipulates that the freestream would follow the Prandtl-Meyer expansion as if the attached viscous layer does not exist, and the fluid within the viscous layer follows a stream tube expansion to the final pressure, one would obtain the velocity profile (Fig. 3) within the viscous layer at the initial section of the constant pressure developing mixing region which is given by

$$\phi_2 = \frac{u_2}{u_{2a}} = \left[\frac{C_{1a}^2}{C_{2a}^2} \left[\phi_1^2 + \left(\frac{C_{2a}^2}{C_{1a}^2} - 1 \right) \frac{1 - C_{1a}^2 \phi_1^2}{1 - C_{1a}^2} \right] \right]^{1/2} \tag{23}$$

whose corresponding location within the viscous layer after the expansion may be determined from the continuity relationship

$$\int_0^{y_2} \frac{dy_2}{\delta_1} = \left(\frac{1-C_{1a}^2}{1-C_{2a}^2} \right)^{1/\gamma-1} \frac{C_{1a}}{C_{2a}} \int_0^{y_1} \frac{\phi_1}{\phi_2} \frac{dy_1}{\delta_1} \tag{24}$$

One now assumes that a linear velocity profile prevails within the subsequent developing constant pressure jet mixing region, and stipulates that at each location along the path of the mixing region, the slope of the profile corresponds to that of the error function profile given by

$$\partial\phi/\partial\eta = 1/\sqrt{\pi} \tag{25}$$

where $\eta = \sigma(y/x)$. Continuity and momentum principles are applied to this region and it can be shown that the velocity of the dividing streamline and the thicknesses of the shear layer above and below the dividing streamline can be determined from

$$\ln \frac{1-C_d^2}{1-C_{2m}^2} = \frac{2C_{2m}}{C_{2a}} \left(\frac{1}{2C_{2m}} \ln \frac{1+C_{2m}}{1-C_{2m}} - 1 \right) \times \left[\int_0^{y_{1m}} \frac{\rho_1}{\rho_{1a}} \phi_1 \mathrm{d} \frac{y_1}{\delta_1} \right] \Big/ \left[\int_0^{y_{1m}} \frac{\rho_1}{\rho_{1a}} \phi_1 \phi_2 \mathrm{d} \frac{y_1}{\delta_1} \right] \tag{26}$$

$$\frac{\delta_a + \delta_b}{\delta_1} = \frac{C_{1a} C_{2a}}{1 - C_{2a}^2} \left[\left(\frac{1-C_{1a}^2}{1-C_{2a}^2} \right)^{1/\gamma-1} \times \int_0^{y_{1m}} \frac{\rho_1}{\rho_{1a}} \phi_1 \phi_2 \, \mathrm{d} \frac{y_1}{\delta_1} \right] \Big/ \left(\frac{1}{2C_{2m}} \ln \frac{1+C_{2m}}{1-C_{2m}} - 1 \right) \tag{27}$$

and the corresponding location along the length of the mixing region is given by

$$\frac{x}{\delta_1} = \frac{\sigma}{\sqrt{\pi}} \frac{C_{2a}}{C_{2m}} \frac{\delta_a + \delta_b}{\delta_1} \tag{28}$$

with

$$C_d / C_{2m} = \delta_b / (\delta_a + \delta_b) \tag{29}$$

the dividing streamline can also be traced throughout this region, and the shear stress along the dividing streamline may be evaluated from

$$\frac{\tau_d}{\rho_{2a} u_{2a}^2} = \frac{1 - C_{2a}^2}{C_{2a}^2} \frac{\mathrm{d}}{\mathrm{d}x} \times \left[\delta_b \left(\frac{1}{2C_d} \ln \frac{1+C_d}{1-C_d} - 1 \right) \right] \tag{30}$$

In all practical calculations, the velocity at the edge of the new mixing layer is always less than the free streamline value (i.e., $C_{2m} < C_{2a}$). An equivalent shear layer thickness above the dividing streamline is determined by extending the velocity profile to the freestream value for the subsequent recompression calculations.

2) Recompression-Reattachment Processes

Calcualtions of recompression up to the point of reattachment follow precisely the scheme developed previously.[9] The system of equations involved with this process will be listed here without detailed explanations.The continuity,streamwise, and transverse momentum equations for the shear layer above the dividing streamline are

$$\tan\beta_e = \frac{v_e}{u_e} = \frac{d}{dx}\left[\delta_a \int_0^1 \left(1-\frac{\rho}{\rho_e}\phi\right) d\zeta\right] - \frac{\delta_a \int_0^1 \frac{\rho}{\rho_e}\phi\, d\zeta}{(1-C_e^2)^{1/\gamma-1} C_e} \frac{d}{dx}\left[(1-C_e^2)^{1/\gamma-1} C_e\right] \tag{31}$$

$$\frac{d}{dx}\left[(1-C_e^2)^{1/\gamma-1} C_e^2 \delta_a \int_0^1 \frac{\rho}{\rho_e}\phi(1-\phi)\, d\zeta\right] + (1-C_e^2)^{1/\gamma-1} C_e \delta_a \int_0^1 \left(1-\frac{\rho}{\rho_e}\phi\right) d\zeta \frac{dC_e}{dx} - \frac{3}{16}\frac{\gamma-1}{\gamma}\frac{d}{dx}\left[(1-C_e^2)^{\gamma/\gamma-1}\delta_a\left(\frac{p_d}{p_e}-1\right)\right] = \frac{\tau_d}{\rho_{1a}u_{1a}^2}(1-C_{1a}^2)^{1/\gamma-1} C_{1a}^2 \tag{32}$$

$$\frac{p_d}{p_e} = 1 + \frac{2\gamma}{\gamma-1}\left\{\frac{C_e^2}{1-C_e^2}\left(\tan^2\beta_e - \tan\beta_e \frac{d\delta_a}{dx}\right) + \frac{1}{(1-C_e^2)^{\gamma/\gamma-1}}\frac{d}{dx}\left[(1-C_e^2)^{1/\gamma-1} C_e^2 \delta_a \int_0^1 \frac{\rho}{\rho_e}\phi^2 \tan\beta\, d\zeta\right]\right\} \tag{33*}$$

The velocity profile for this upper layer is given by

$$\phi = \phi_d + \frac{\partial\phi}{\partial\zeta}\Big|_d \zeta + \left[3(1-\phi_d) - 2\frac{\partial\phi}{\partial\zeta}\Big|_d\right]\zeta^2 + \left[\frac{\partial\phi}{\partial\zeta}\Big|_d - 2(1-\phi_d)\right]\zeta^3$$

where $(\partial\phi/\partial\zeta)|_d$ and ϕ_d are proportional to each other; the constant of proportionality is evaluated from the information at the end of the mixing region.

The continuity, streamwise, and transverse momentum relations for the shear layer below the dividing streamline and the backflow are given by

$$\frac{\delta_b}{h_b} = \frac{4}{\pi}\frac{p_w}{p_d}\frac{C_d}{(1-C_b^2)^{1/2}}\tan^{-1}\frac{C_b}{(1-C_b^2)^{1/2}} / [-\ln(1-C_d^2)] \tag{34}$$

$$\frac{\gamma-1}{2\gamma}\frac{d}{dx}\left[\frac{p_d}{p_e}\frac{p_e}{p_{o_\infty}}\left(\delta_b + \frac{p_w}{p_d}h_b\cos\theta_r\right)\right] + \frac{\gamma-1}{2\gamma}\sin\theta_r\frac{\cos(\theta_r-\theta_\ell)}{\cos\theta_\ell}\cdot\frac{p_w}{p_d}\cdot\frac{p_d}{p_e}\frac{p_e}{p_{o_\infty}}$$
$$+\frac{d}{dx}\left\{\frac{p_d}{p_e}\frac{p_e}{p_{o_\infty}}\left[\delta_b\left(\frac{1}{2C_d}\ln\frac{1+C_d}{1-C_d}-1\right) + \frac{p_w}{p_d}h_b\cos\theta_r\cdot\left((1-C_b^2)^{-1/2}-1\right)\right]\right\} = \frac{\tau_d}{\rho_{1a}u_{1a}^2}(1-C_{1a}^2)^{1/\gamma-1} C_{1a}^2 \tag{35}$$

and

$$\frac{p_w}{p_d} = 1 + \frac{2\gamma}{\gamma-1}\frac{\sin\theta_r}{(p_d/p_e)(p_e/p_{o_\infty})}\left\{\frac{\cos\theta_\ell}{\cos(\theta_r-\theta_\ell)}\frac{\tau_d}{\rho_{1a}u_{1a}^2}\right.$$
$$\left.\cdot(1-C_{1a}^2)^{1/\gamma-1}C_{1a}^2 - \frac{d}{dx_w}\left[\frac{p_d}{p_e}\frac{p_e}{p_{o_\infty}}\delta_b\left(\frac{1}{2C_d}\ln\frac{1+C_d}{1-C_d}-1\right)\right]\right\} \tag{36}$$

where a linear velocity profile and a cosine profile are assumed for the lower viscous layer and the backflow.

3) Calculations of Mixing and Recompression Processes

For a given approaching flow condition (i.e., M_1 and δ_1^{**}/H), a pair of values of p_b/p_1 and ℓ_m/H are selected; ℓ_m being the length of the constant pressure mixing region along the jet boundary and also identifying the beginning of the recompression region. Calculations may be carried out for the expansion, mixing processes, and the system of equations for recompression process may be integrated with the initial conditions provided from the end of the mixing region. At the point of reattachment, the wall pressure can be determined from wake momentum equation [Eq. (35)] and its freestream condition as well as the thickness of the viscous layer are determined from the continuity and normal momentum equations [Eqs. (31) and (33)]. A residue is evaluated from Eq. (32) and the value of ℓ_m/H corresponding to the selected base pressure ratio p_b/p_1 must be iterated upon until this residue at the point of reattachment vanishes. The correct combinations of p_b/p_1 and ℓ_m/h as the solution of the problem must, however, be determined from the considerations of redevelopment.

It should be mentioned that for all results reported here, the value of σ is 12, and the expression of the eddy diffusivity along the dividing streamline for evaluation of the shear stress within the recompression region is given by

$$(\epsilon/\epsilon_m) = (u_e/u_{em})[(x_r+\ell_m)/\ell_m]^4 \tag{37}$$

*Here, β or β_e refer to the local streamline angle with respect to the dividing streamline.

where the subscript m refers to the section at the end of the mixing region and x_r represents the length along the recompression region.

4) Property Relations between Sections **R** *and* **R′** *and the Matching with the Redevelopment Calculations*

After the flow reattaches (Fig. 1) at section R, the viscous layer will go through a triangular region until section R' is reached. It is necessary to relate the flow properties of these two sections (R and R') which have the same freestream condition. The geometric configuration (see Fig. 4) obviously yields

$$\delta_{R'} = \delta_R \cos\theta_r, \qquad \Delta x_w = \delta_R \sin\theta_r \tag{38}$$

The continuity principle would also relate the mass integrals of the two sections by

$$\cos\theta_r \int_0^1 \frac{\rho}{\rho_e}\phi_{R'}\,\mathrm{d}\zeta = \int_0^1 \frac{\rho}{\rho_e}\phi_R\,\mathrm{d}\zeta \tag{39}$$

where

$$\phi_{R'} = S_{R'}\zeta + (3-2S_{R'})\zeta^2 + (S_{R'}-2)\zeta^3$$

and $\phi_R = 3\zeta^2 - 2\zeta^3$, and the slope parameter $S_{R'}$ may be determined from this relationship.

Upon applying the momentum principle to the triangular region normal to the wall (Fig. 4) and neglecting the difference of the lateral shear stresses from sections R and R', one may evaluate the pressure difference across the viscous layer at section R' from

$$\frac{p_w}{p_e}\Big|_{R'} = 2\Big[\frac{5}{8} - \frac{1}{8}\frac{p_w}{p_e}\Big|_R + \frac{2\gamma C_e^2}{(\gamma-1)(1-C_e^2)} \cdot \Big\{\cot\theta_r \tan\beta_e \int_0^1 \frac{\rho}{\rho_e}\phi^2(2\zeta-\zeta^2)\,\mathrm{d}\zeta\Big|_{R'} + \int_0^1 \frac{\rho}{\rho_e}\phi^2[1-\cot\theta_r\tan(\theta_r+\beta_e)(2\zeta-\zeta^2)]\,\mathrm{d}\zeta\Big|_R\Big\}\Big] \tag{40}$$

Thus, for each pair of values of p_b/p_1 and ℓ_m/H that satisfies the condition at the point of reattachment, they would yield a unique set of information at R' ($C_{er'}$, $p_w/p_{eR'}$, $S_{R'}$, etc.). The particular constant of proportionality which will be needed for the redevelopment calculations may be evaluated according to Eq. (13). It is now obvious that the correct pair of values of p_b/p_1 and ℓ_m/H for the problem would produce the same p_w/p_e value at R' when the freestream Crocco number assumes the value of $C_{eR'}$ from the redevelopment calculations. A typical set of calculations illustrating the matching of the flowfields and yielding solutions to the problem is given in Fig. 5.

Results of Calculations

Extensive numerical calculations have been carried out for $M_1=2$ and $M_1=1.5$ with a wide range of intial momentum thickness ratio δ_1^{**}/H. The resulting base pressure ratio p_b/p_1 are plotted in Figs. 6 and 7 for comparison with the experimental data from various sources.[15-16] (These data are directly adopted from Ref. 5.) Results produced by Alber and Lees[5] are plotted in the same figures. The detailed pressure distribution for a particular flow case illustrating the recompression, reattachment, and redevelopment processes are presented in Fig. 8. Other interesting features, such as the energizing and de-energizing aspects of the flow along dividing streamline, can easily be extracted from the results of calculations and they are not presented here.

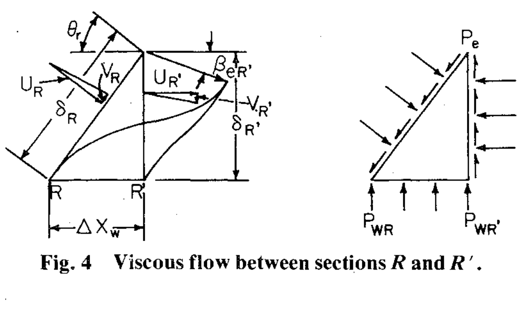

Fig. 4 Viscous flow between sections R and R'.

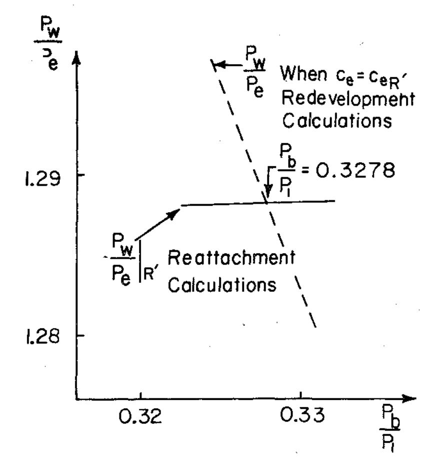

Fig. 5 Matching of p_w/p_e from reattachment and redevelopment calculations. $\delta_1^{**}/H=0.01, M_1=2.0$.

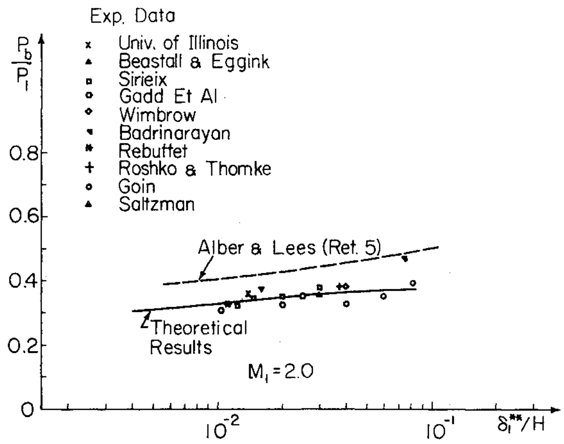

Fig. 6 Base pressure ratio as influenced by momentum thickness of approaching flow ($M_1=2.0$).

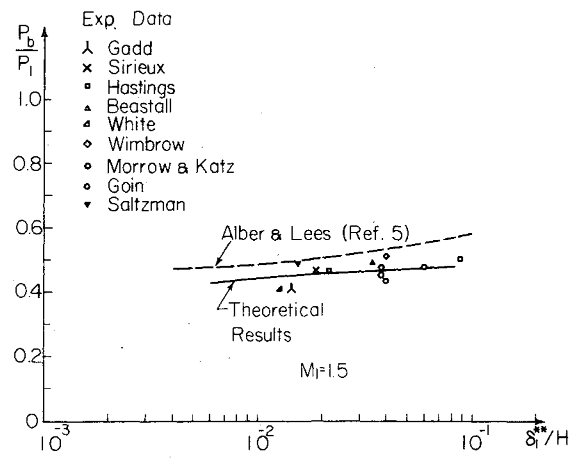

Fig. 7 Base pressure ratio as influenced by momentum thickness of approaching flow ($M_1=1.5$).

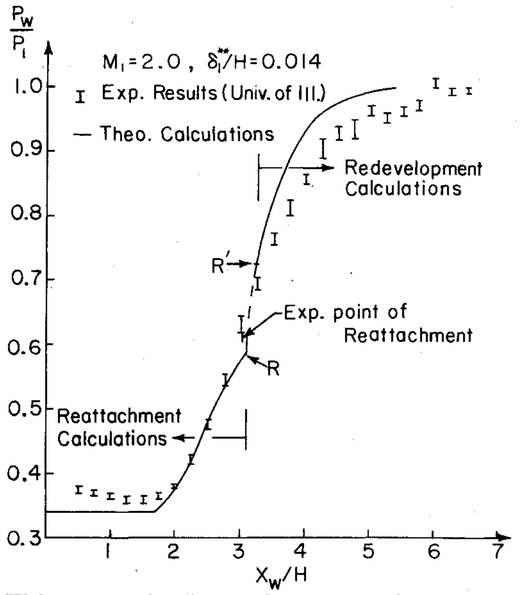

Fig. 8 Wall pressure distribution downstream of a backstep ($M_1 = 2.0$, $\delta_1^{**}/H = 0.014$).

Discussion

It should be mentioned that in the component approach of studying separated flows, the data produced from the upstream flow component provide the initial conditions for the subsequent flow analysis and in such detailed calculations one cannot expect smooth transition between various flow regions throughout the flow. In addition, due to our lack of information of turbulence, especially within the recompression and redevelopment regions, estimation of eddy diffusivity along the dividing streamline according to Eq. (37) can only be based on speculation. Fortunately, this same expression is applied for both cases of $M_1 = 2$ and $M_1 = 1.5$ throughout a wide range of the initial momentum thickness ratio and reasonably good results have been obtained. Hopefully, future research on the turbulent structure within the recompression—reattachment and redevelopment flows would improve the capability of performing detailed calculations for these regions.

The most important goal of this study is to examine the possibility of interpreting the flow redevelopment as a process of relaxation of the pressure difference across the viscous layer within the supersonic flow regime. The analysis and the derived results fully supported such an interpretation. In addition, since the submission of the original manuscript of this paper, additional research effort has been performed to examine the supersonic flow past a backstep in axisymmetric configuration.[27] It has been found that the concept of relaxation of pressure difference can indeed be adopted to predict the base pressure and to describe many special features associated with axisymmetric flows (e.g., effect of sting, overshoot in static pressure on the sting).

It may be concluded that: 1) The redevelopment after flow reattachment is an equally important flow component in the Chapman-Korst model. 2) The flow redevelopment process may be interpreted as relaxation of the pressure difference across the viscous layer. 3) Upon adopting such an interpretation, the fully rehabilitated state is a saddle-point singularity of the system of equations which provides the closure condition for the Chapman-Korst model.

Appendix: The Asymptotic State as a Saddle-Point Singularity

Equation (19b) is written as

$$\frac{\mathrm{d}(p_w/p_e)}{\mathrm{d}\,C_e} = \frac{D_3(A_1B_2 - A_2B_1) + D_1(B_1C_2 - B_2C_1)}{D_3(A_3B_1 - A_1B_3) + D_1(B_3C_1 - B_1C_3)} \qquad \text{(A1)}$$

where

$$D_1 = \tan\beta_e, \quad D_3 = \frac{\gamma - 1}{2\gamma}\left(\frac{p_w}{p_e} - 1\right) - \frac{C_e^2}{1 - C_e^2}\tan^2\beta_e$$

Since $D_1 = 0$ and $D_3 = 0$ at the asymptotic state, it is a singularity of the equation. If one integrates Eq. (A1) toward downstream with slightly different conditions, widely different values and divergent trends are observed. This suggests that the asymptotic state is a saddle-point singularity. To establish its true behavior in the vicinity of the asymptotic state, detailed examination is required.

Upon rewriting Eq. (A1) as

$$\frac{\mathrm{d}(p_w/p_e)}{\mathrm{d}C_e} = \frac{p\,D_3 + qD_1}{rD_3 + sD_1}$$

and restricting to small deviations from the asymptotic state, one obtains the linearized version of Eq. (A2) as

$$\mathrm{d}y/\mathrm{d}x = (tx + uy)/(vx + wy)$$

where

$$t = -\frac{q_\infty}{C_{e_\infty}}\left[\frac{2}{\gamma - 1}\,\frac{C_{e_\infty}^2}{1 - C_{e_\infty}^2} - 1\right]^{1/2}$$

$$u = [(\gamma - 1)/2\gamma]p_\infty$$

$$v = \frac{s_\infty}{C_{e_\infty}}\left[\frac{2}{\gamma - 1}\,\frac{C_{e_\infty}^2}{1 - C_{e_\infty}^2} - 1\right]^{1/2}$$

$$w = [(\gamma - 1)/2\gamma)r_\infty$$

where

$$y = [(p_w/p_e) - 1] \qquad x = C_e - C_{e_\infty}$$

$p_\infty, q_\infty, r_\infty$, and s_∞ are their respective functional values at the asymptotic condition, and the relationship

$$C_e - C_{e_\infty} = -\frac{C_{e_\infty}\,\beta_e}{\left[\frac{2}{\gamma - 1}\,\frac{C_{e_\infty}^2}{1 - C_{e_\infty}^2} - 1\right]^{1/2}}$$

has been introduced

Since $p_\infty, q_\infty, r_\infty$, and s_∞ are complicated functions of the flow properties and the partial derivatives of the integral quantities at the asymptotic state, considerable effort would be needed to establish their values and their trends. Instead, straight forward numerical evaluations are employed, and the results show that the quantity $(uv - tw)$ is negative for all supersonic Mach numbers (up to $M_{\infty\omega\vartheta} = 10$), which is the necessary and sufficient condition for a saddle-point singularity.

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